CDI-II – Prática F5 7/4/21

1

Ficha 5

1 – Trivial.

2 – $$V(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$(x,y,z) \xmapsto{r} \sqrt{x^2+y^2+z^2} \xmapsto{f} \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "r"] \arrow[rr, bend right, "V = f\circ r"'] & \mathbb{R} \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$$V(x,y,z) = f\left(r(x,y,z)\right)$$

$$f(r) = \frac{1}{r} \longrightarrow f'(r) = -\frac{1}{r^2}$$

$$r(x,y,z) = \left(x^2+y^2+z^2\right)^{1/2}$$

$$V(x,y,z) = f\big(r(x,y,z)\big)$$ Regra da Cadeia

$$\frac{\partial V}{\partial x} = f'(r)\,\frac{\partial r}{\partial x} \qquad f'(r) = \frac{df}{dr}(r)$$

$$f'(r) = -\frac{1}{r^2}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2}\left(x^2+y^2+z^2\right)^{-\frac{1}{2}} 2x$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$$\frac{\partial V}{\partial x} = -\frac{1}{r^2}\frac{x}{r} = -\frac{x}{r^3}$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{x}{r^3}\right)$$

$$= -\frac{r^3 - x\,3r^2 \frac{\partial r}{\partial x}}{r^6} \qquad = \frac{x}{r}$$

$$\frac{\partial}{\partial x}\left(r^3(x,y,z)\right) = 3r^2 \frac{\partial r}{\partial x}$$

$$(x,y,z) \longmapsto r(x,y,z) \longmapsto r^3(x,y,z)$$

$$\frac{d}{dx}\left(u^3(x)\right) = 3u^2 u'(x)$$

$$\frac{\partial^2 V}{\partial x^2} = -\frac{r^3 - 3x^2 r}{r^6} = \frac{3x^2 - r^2}{r^5}$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{3y^2 - r^2}{r^5}$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{3z^2 - r^2}{r^5}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad //$$

<u>Note:</u> $\frac{\partial V}{\partial x} = f'(r)\frac{\partial r}{\partial x}$

$$\frac{\partial V}{\partial x}(x,y,z) = f'\big(r(x,y,z)\big)\frac{\partial r}{\partial x}(x,y,z)$$

$$\frac{\partial^2 V}{\partial x^2} = f''(r)\underbrace{\frac{\partial r}{\partial x}}_{\frac{x}{r}}\underbrace{\frac{\partial r}{\partial x}}_{\frac{x}{r}} + f'(r)\frac{\partial^2 r}{\partial x^2} \; ?$$

etc. . . .

4

$$3 - w(x,y) = f(y-x,\; x+y) =$$

$$w(x,y) = f\big(u(x,y),\; v(x,y)\big)$$

$$\begin{cases} u(x,y) = y - x \\ v(x,y) = x + y \end{cases} \qquad f : \mathbb{R}^2 \to \mathbb{R},\ C^2$$

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}^{=-1} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}^{=1}$$

$$\frac{\partial w}{\partial x}(x,y) = -\frac{\partial f}{\partial u}\big(u(x,y), v(x,y)\big) + \frac{\partial f}{\partial v}\big(u(x,y), v(x,y)\big)$$

$$\frac{\partial^2 w}{\partial x^2} = -\left(\frac{\partial^2 f}{\partial u^2}\overset{-1}{\frac{\partial u}{\partial x}} + \frac{\partial^2 f}{\partial v \partial u}\overset{1}{\frac{\partial v}{\partial x}}\right) +$$

$$+ \left(\frac{\partial^2 f}{\partial u \partial v}\overset{-1}{\frac{\partial u}{\partial x}} + \frac{\partial^2 f}{\partial v^2}\overset{1}{\frac{\partial v}{\partial x}}\right)$$

$$\frac{\partial^2 w}{\partial x^2} = + \cancel{\frac{\partial^2 f}{\partial u^2}} \overset{+}{-} \frac{\partial^2 f}{\partial v \partial u} \overset{+}{-} \frac{\partial^2 f}{\partial u \partial v} + \cancel{\frac{\partial^2 f}{\partial v^2}}$$

...

$$\frac{\partial^2 w}{\partial y^2} = \cancel{\frac{\partial^2 f}{\partial u^2}} + \frac{\partial^2 f}{\partial v \partial u} + \frac{\partial^2 f}{\partial u \partial v} + \cancel{\frac{\partial^2 f}{\partial v^2}}$$

4 — **Nota:** $\begin{cases} \det Hf(a) = \lambda_1 \times \cdots \times \lambda_n \\ \mathrm{Tr}\, Hf(a) = \lambda_1 + \cdots + \lambda_n \end{cases}$

$\mathbb{R}^2$: 2 eq., 2 incógnitas

4-e) $\begin{cases} \dfrac{\partial f}{\partial x} = z - 2x = 0 \\ \dfrac{\partial f}{\partial y} = -2y = 0 \\ \dfrac{\partial f}{\partial z} = x = 0 \end{cases}$

$(0,0,0)$ único ponto crítico de f.

$$Hf(0,0,0) = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}_{3\times 3}$$

v.p. $\lambda_1, \lambda_2, \lambda_3$:

$$\det\left(Hf(0,0,0) - \lambda I\right) = 0$$

$$\det \begin{bmatrix} -2-\lambda & 0 & 1 \\ 0 & -2-\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix} = 0$$

Laplace

$$(-2-\lambda) \det \begin{bmatrix} -2-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = 0$$

$$(-2-\lambda)\left((-2-\lambda)(-\lambda) - 1\right) = 0$$

$$-2-\lambda=0 \quad \vee \quad (2+\lambda)\lambda-1=0$$

$$\boxed{\lambda=-2} \quad \vee \quad \underbrace{\lambda^2+2\lambda-1=0}$$

$$\lambda=\frac{-2\pm\sqrt{4+4}}{2}$$

$$\lambda=\frac{-2\pm 2\sqrt{2}}{2}$$

$$\lambda=-1\pm\sqrt{2}$$

$$\underbrace{\lambda_1=-2 \; ; \; \lambda_2=-1-\sqrt{2}}_{<0} \; ; \; \underbrace{\lambda_3=-1+\sqrt{2}}_{>0}$$

$(0,0,0)$ é p.^{to} crítico de f mas
não é p.^{to} de extremo de
f.

— // —

4-f) $\left\{\begin{array}{l} \dfrac{\partial f}{\partial x} = 3x^2 = 0 \leftarrow \\ \dfrac{\partial f}{\partial y} = -4y^3 = 0 \leftarrow \end{array}\right.$

$(0,0)$ único p.^{to} crítico de f.

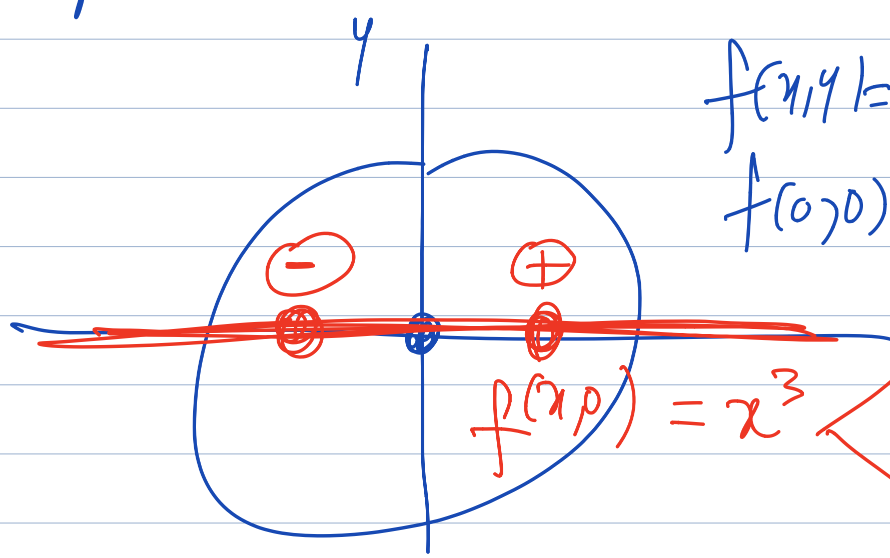
$$Hf(x,y) = \begin{bmatrix} 6x & 0 \\ 0 & -12y^2 \end{bmatrix}$$

$$Hf(0,0)=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \qquad \begin{aligned}\lambda_1&=0\\ \lambda_2&=0\end{aligned}$$

Análise local de f "perto" do ponto crítico $(0,0)$.

$f(x,y)=x^3-y^4$

$f(0,0)=0$



$f(x,0)=x^3$ $>0,\ x>0$; $<0,\ x<0$

sinal de $f(x,y)-f(0,0)$

sinal de $f(x,y)$

$(0,0)$ não é p.to de extremo de f.